Two cities A and B lie on the equator such that their longitudes are $65^\circ E$ and $35^\circ W$, respectively. Find the distance between the two cities, provided the radius of the Earth is $6400\,\mathrm{km}$.

Solution:

The formula to calculate the distance between two points on the Earth's surface along the equator is:

$$\text{Distance} = \theta \cdot R$$

where:

- θ is the angular difference in radians, and
- R is the radius of the Earth.

Step 1: Calculate the angular difference θ

The total longitude difference is:

$$\theta = 65^\circ + 35^\circ = 100^\circ$$

Convert θ to radians:

$$\theta\,(\text{in radians}) = \theta \cdot \frac{\pi}{180} = 100 \cdot \frac{\pi}{180} = \frac{5\pi}{9}\ \text{radians.}$$

Step 2: Calculate the distance

Using the given radius $R = 6400\,\mathrm{km}$:

$$\text{Distance} = \theta \cdot R = \frac{5\pi}{9} \cdot 6400$$

$$\text{Distance} = \theta \cdot R = \frac{5\pi}{9} \cdot 6400$$

$$\text{Distance} \approx 11185.3\,\text{km}.$$

Answer:

The distance between the two cities is approximately 11,185.3 km.

1. زاویہ کو سادہ شکل میں لائیں:

$\frac{207\pi}{2} - \theta$

یہ ایک بڑا زاویہ ہے، اس کو 2π کی بار بار تقسیم سے بنیادی دائرے (0 سے 2π) کے اندر لائیں:

$$: (4\pi/2 = \text{Divide } 207 \text{ by } 4 \text{ (as } 2\pi$$

$$\text{remainder } 3 \; 51 = 4 \div 207$$

لہذا، باقی زاویہ:

$$\frac{3\pi}{2} - \theta$$

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2. بنیادی زاویہ کی شناخت کریں:

$\frac{3\pi}{2} - \theta$ ایک معیاری زاویہ ہے جو 270° کے برابر ہے۔ یہ زاویہ چوتھے کوارڈینٹ میں موجود ہے۔

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3. ٹرگنومیٹرک فنکشنز کی ویلیوز معلوم کریں:

- $:\sin\theta$

$$1- = \frac{3\pi}{2}\sin$$

- $\tan\theta$:

$$\tan\frac{3\pi}{2} = \frac{\sin\theta}{\cos\theta} = \frac{-1}{0}$$

یہ غیر معین (undefined) ہے، کیونکہ $\cos\theta = 0$.

- $\cot\theta$:

$$\cot\frac{3\pi}{2} = \frac{\cos\theta}{\sin\theta} = \frac{0}{-1} = 0$$

- $\sec\theta$:

$$\sec\frac{3\pi}{2} = \frac{1}{\cos\theta} = \frac{1}{0}$$

یہ بھی غیر معین (undefined) ہے۔

- $\csc\theta$:

$$\csc\frac{3\pi}{2} = \frac{1}{\sin\theta} = \frac{1}{-1} = -1$$

حتمی جواب:

$\sin\frac{207\pi}{2} = -1$, $\cos\frac{207\pi}{2} = 0$, $\tan\frac{207\pi}{2}$ is undefined,

$\cot\frac{207\pi}{2} = 0$, csc $\frac{207\pi}{2}$ is undefined, sec $\frac{207\pi}{2} = -1$.

If $\cos\alpha = \frac{3}{5}$, find the values of $\tan 2\alpha$ for:

1. $0 < \alpha < \frac{\pi}{2}$, and
2. $\pi < \alpha < \frac{3\pi}{2}$.

Solution:

Step 1: Recall the formula for $\tan 2\alpha$:

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}.$$

To use this formula, we need to calculate $\tan\alpha$.

Step 2: Find $\sin\alpha$ and $\tan\alpha$

From $\cos\alpha = \frac{3}{5}$, use the Pythagorean identity:

$$\sin^2\alpha + \cos^2\alpha = 1.$$

$$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}.$$

$$\sin\alpha = \pm\frac{4}{5}.$$

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- For $0 < \alpha < \frac{\pi}{2}$ (first quadrant): $\sin\alpha = \frac{4}{5}$.
- For $\pi < \alpha < \frac{3\pi}{2}$ (third quadrant): $\sin\alpha = -\frac{4}{5}$.

Now, calculate $\tan\alpha$:

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}.$$

- For $0 < \alpha < \frac{\pi}{2}$:

$$\tan\alpha = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}.$$

- For $\pi < \alpha < \frac{3\pi}{2}$:

$$\tan\alpha = \frac{-\frac{4}{5}}{\frac{-3}{5}} = \frac{4}{3}.$$

Step 3: Compute $\tan 2\alpha$

Using $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$:

1. For $0 < \alpha < \frac{\pi}{2}$:

1. For $0 < \alpha < \frac{\pi}{2}$:

$$\tan 2\alpha = \frac{2 \cdot \frac{4}{3}}{1 - \left(\frac{4}{3}\right)^2}.$$

$$\tan 2\alpha = \frac{\frac{8}{3}}{1 - \frac{16}{9}} = \frac{\frac{8}{3}}{-\frac{7}{9}} = \frac{8}{3} \cdot \frac{-9}{7} = -\frac{24}{7}.$$

2. For $\pi < \alpha < \frac{3\pi}{2}$:

$$\tan 2\alpha = \frac{2 \cdot \frac{4}{3}}{1 - \left(\frac{4}{3}\right)^2}.$$

The calculation is the same because $\tan \alpha = \frac{4}{3}$, so:

$$\tan 2\alpha = -\frac{24}{7}.$$

Answer:

- For $0 < \alpha < \frac{\pi}{2}$: $\tan 2\alpha = -\frac{24}{7}$.
- For $\pi < \alpha < \frac{3\pi}{2}$: $\tan 2\alpha = -\frac{24}{7}$.

Draw the graph of $f(x) = 6\csc\left(\frac{\pi x}{3} + \pi\right)$.

Solution:

Step 1: Rewrite the function for better understanding

The given function is:

$$f(x) = 6\csc\left(\frac{\pi x}{3} + \pi\right).$$

We know that $\csc(x) = \frac{1}{\sin(x)}$, so the graph of $f(x)$ depends on the behavior of the sine function.

Step 2: Analyze the sine function

The argument of the cosecant function is:

$$\frac{\pi x}{3} + \pi.$$

This is a horizontally stretched sine function shifted by π.

Key points to consider:

1. The period of $\sin\left(\frac{\pi x}{3} + \pi\right)$ is determined by the coefficient of x:

$$\text{Period} = \frac{2\pi}{\frac{\pi}{3}} = 6.$$

   Hence, the graph of $f(x)$ repeats every 6 units.

2. The vertical scaling factor is 6, meaning the cosecant will have asymptotes where the sine function is zero, with maximum and minimum distances scaled by 6.

Step 3: Identify key points for $\sin\left(\frac{\pi x}{3} + \pi\right)$

Let's find where the sine function is zero, 1, or -1:

- **Zeros:**
  Solve $\frac{\pi x}{3} + \pi = n\pi$, where $n \in \mathbb{Z}$:

$$\frac{\pi x}{3} = n\pi - \pi, \quad x = 3n - 3.$$

  Thus, zeros occur at $x = 3n - 3$ (e.g., $x = -3, 0, 3, 6, \ldots$).

- **Maxima and Minima:**
  The sine function reaches its maximum or minimum values (± 1) at:

$$\frac{\pi x}{3} + \pi = \frac{\pi}{2} + 2n\pi \quad \text{(for maxima)},$$
$$\frac{\pi x}{3} + \pi = -\frac{\pi}{2} + 2n\pi \quad \text{(for minima)}.$$

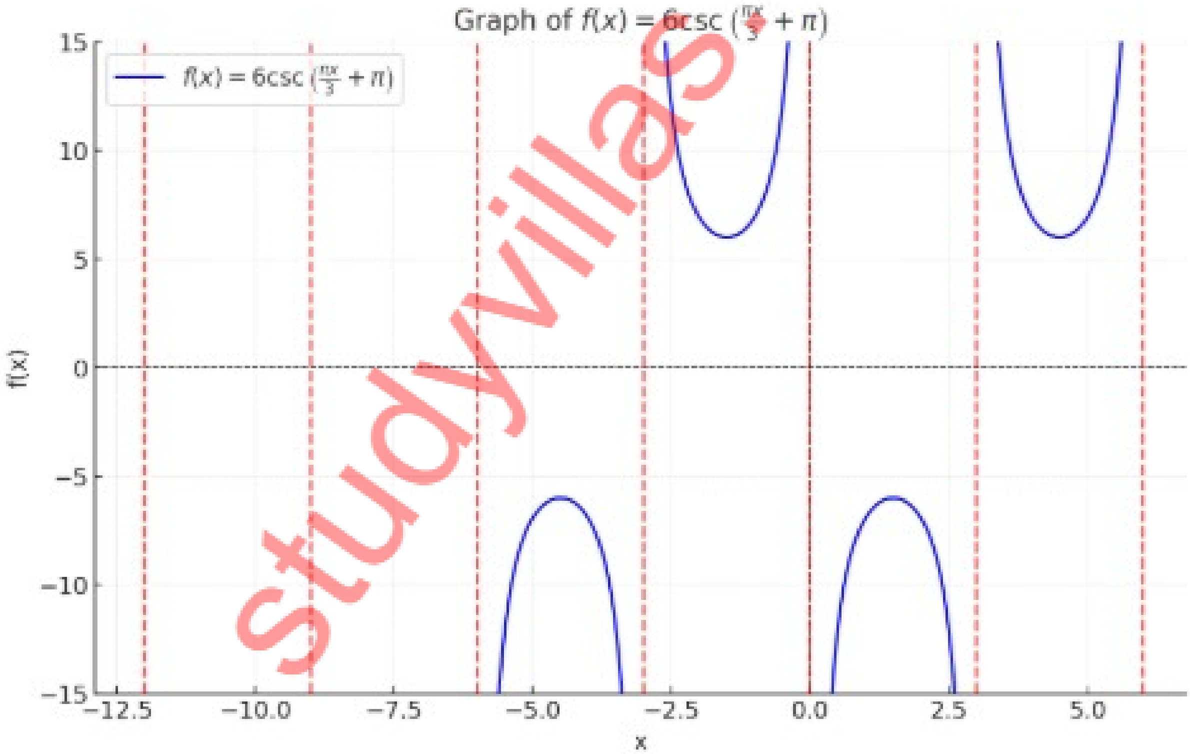
  Solving gives the respective x-values for peaks and troughs.

Step 4: Draw the graph

The graph of $f(x) = 6\csc\left(\frac{\pi x}{3} + \pi\right)$ will:

- Have vertical asymptotes at $x = -3, 0, 3, 6, \ldots,$
- Have the amplitude scaled by 6, so the graph reaches 6 and -6 at the maxima and minima, respectively.

I'll create the graph. Give me a moment.



Here is the graph of $f(x) = 6\csc\left(\frac{\pi x}{3} + \pi\right)$:

- The blue curve represents $f(x)$, with peaks and troughs scaled by a factor of 6.
- The red dashed lines indicate vertical asymptotes where $\sin\left(\frac{\pi x}{3}+\pi\right)=0$, occurring at $x=-3, 0, 3, 6, \ldots$.
- The graph is periodic with a period of 6.

Prove that:

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right).$$

Solution:

Step 1: Recall the trigonometric identity

The standard trigonometric identity for the difference of two sines is:

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$$

In this case, let $A = \alpha$ and $B = \beta$. Rewriting the identity:

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right).$$

Step 2: Expand $\sin\alpha$ and $\sin\beta$

Using the angle sum and difference formulas for sine:

$$\sin\alpha = \sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right),$$

$$\sin\beta = \sin\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right).$$

Step 3: Apply the sine addition and subtraction formulas

The sine addition formula states:

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y.$$

For $\sin\alpha$:

$$\sin\alpha = \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).$$

For $\sin\beta$:

$$\sin\beta = \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).$$

Step 4: Subtract $\sin\beta$ from $\sin\alpha$

$$\sin\alpha - \sin\beta = \left[\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right]$$

$$-\left[\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right].$$

Simplify:

$$\sin\alpha-\sin\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).$$

Answer:

$$\sin\alpha-\sin\beta=2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right).$$

This proves the given identity.

Solve the equation:

$$9 \sin w - 2 = 4 \sin^2 w.$$

Find the solutions for w in the interval $0 \leq w \leq 2\pi$.

Solution:

Step 1: Rewrite the equation

The given equation is:

$$9 \sin w - 2 = 4 \sin^2 w.$$

Rearrange terms to form a standard quadratic equation in $\sin w$:

$$4 \sin^2 w - 9 \sin w + 2 = 0.$$

Step 2: Solve the quadratic equation

Let $x = \sin w$. The equation becomes:

$$4x^2 - 9x + 2 = 0.$$

Solve this quadratic equation using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a = 4$, $b = -9$, and $c = 2$.

Substitute the values:

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(4)(2)}}{2(4)} = \frac{9 \pm \sqrt{81 - 32}}{8}.$$

$$x = \frac{9 \pm \sqrt{49}}{8}.$$

$$x = \frac{9 \pm 7}{8}.$$

Thus, the two solutions for x are:

$$x = \frac{9 + 7}{8} = 2 \quad \text{and} \quad x = \frac{9 - 7}{8} = \frac{1}{4}.$$

Step 3: Check the solutions

The sine function satisfies $-1 \leq \sin w \leq 1$. Therefore, $x = 2$ is invalid since it lies outside this range.

The valid solution is:

$$\sin w = \frac{1}{4}.$$

Step 4: Solve for w

Using $\sin w = \frac{1}{4}$, find w in the interval $0 \leq w \leq 2\pi$:

$$w = \arcsin\left(\frac{1}{4}\right).$$

1. In the first quadrant:

$$w_1 = \arcsin\left(\frac{1}{4}\right).$$

2. In the second quadrant ($\pi - w_1$):

$$w_2 = \pi - \arcsin\left(\frac{1}{4}\right).$$

Using a calculator:

$$w_1 \approx 0.2527 \quad \text{(in radians)},$$
$$w_2 \approx \pi - 0.2527 \approx 2.8889 \quad \text{(in radians)}.$$

Answer:

The solutions for w are:

$$w \approx 0.2527 \quad \text{and} \quad w \approx 2.8889 \quad \text{(in radians)}.$$

Find the exact solutions on the interval $0 \leq x \leq 2\pi$ for the equation:

$$\frac{1}{\sec^2 x} + 2 + \sin^2 x + 4\cos^2 x = 0.$$

Solution:

Step 1: Simplify the equation

Recall that $\sec^2 x = \frac{1}{\cos^2 x}$. Therefore,

$$\frac{1}{\sec^2 x} = \cos^2 x.$$

Substitute this into the equation:

$$\cos^2 x + 2 + \sin^2 x + 4\cos^2 x = 0.$$

Step 2: Use the Pythagorean identity

The Pythagorean identity is $\sin^2 x + \cos^2 x = 1$. Replace $\sin^2 x + \cos^2 x$ with 1:

$$1 + \cos^2 x + 4\cos^2 x + 2 = 0.$$

Simplify:

$$1 + 2 + 5\cos^2 x = 0.$$

$$5\cos^2 x + 3 = 0.$$

Step 3: Solve for $\cos^2 x$

Rearrange the equation:

$$5\cos^2 x = -3.$$

Divide through by 5:

$$\cos^2 x = -\frac{3}{5}.$$

Step 4: Check for solutions

The equation $\cos^2 x = -\frac{3}{5}$ is not possible because $\cos^2 x \geq 0$ for all real x.

Answer:

There are no solutions for the given equation on the interval $0 \leq x \leq 2\pi$.

Simplify the expression:

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + \tan^{-1}(\sqrt{3}).$$

Solution:

Step 1: Evaluate $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

We know that:

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

Thus:

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

Step 2: Evaluate $\tan^{-1}(\sqrt{3})$

We know that:

$$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}.$$

Thus:

$$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}.$$

Step 3: Add the results

Add the two inverse trigonometric values:

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + \tan^{-1}(\sqrt{3}) = \frac{\pi}{4} + \frac{\pi}{3}.$$

Find a common denominator:

$$\frac{\pi}{4} + \frac{\pi}{3} = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{7\pi}{12}.$$

Answer:

The simplified result is:

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + \tan^{-1}(\sqrt{3}) = \frac{7\pi}{12}.$$

$$\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x} = \sec x$$

Solution:

1. **Simplify the first term $\frac{\sin 2x}{\sin x}$:**
   Using the double-angle identity:

$$\sin 2x = 2\sin x\cos x$$

   Substituting into the equation:

$$\frac{\sin 2x}{\sin x} = \frac{2\sin x\cos x}{\sin x} = 2\cos x$$

2. **Simplify the second term $\frac{\cos 2x}{\cos x}$:**
   Using the double-angle identity:

$$\cos 2x = \cos^2 x - \sin^2 x$$

   Substituting into the equation:

$$\frac{\cos 2x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\cos x} = \cos x - \frac{\sin^2 x}{\cos x}$$

3. **Substitute these results into the original equation:**

$$\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x} = (2\cos x) - \left(\cos x - \frac{\sin^2 x}{\cos x}\right)$$

   Simplify the terms:

$$= 2\cos x - \cos x + \frac{\sin^2 x}{\cos x}$$

$$= \cos x + \frac{\sin^2 x}{\cos x}$$

4. Combine into a single fraction:

$$= \frac{\cos^2 x + \sin^2 x}{\cos x}$$

5. Use the Pythagorean identity:

$$\cos^2 x + \sin^2 x = 1$$

   Substituting:

$$= \frac{1}{\cos x}$$

6. Simplify the result:

$$\frac{1}{\cos x} = \sec x$$

Hence proved:

$$\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x} = \sec x$$

Q.5 (b): Find $\tan\frac{\theta}{2}$, given $\cos\theta = \frac{7}{25}$ and $\frac{3\pi}{2} < \theta < 2\pi$.

Solution:

1. Determine the quadrant of θ:
   - $\frac{3\pi}{2} < \theta < 2\pi$ means θ is in the fourth quadrant.
   - In the fourth quadrant:
     - $\cos\theta > 0$
     - $\sin\theta < 0$
2. Find $\sin\theta$ using the Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1$$

Substituting $\cos\theta = \frac{7}{25}$:

$$\sin^2\theta + \left(\frac{7}{25}\right)^2 = 1$$

$$\sin^2\theta + \frac{49}{625} = 1$$

$$\sin^2\theta = 1 - \frac{49}{625} = \frac{625}{625} - \frac{49}{625} = \frac{576}{625}$$

$$\sin\theta = -\frac{\sqrt{576}}{25} = -\frac{24}{25}$$

(Negative because $\sin\theta < 0$ in the fourth quadrant.)

3. **Find $\tan\frac{\theta}{2}$:**

   Use the half-angle identity:

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$$

   Substituting values:

$$\tan\frac{\theta}{2} = \frac{-\frac{24}{25}}{1+\frac{7}{25}}$$

   Simplify the denominator:

$$1+\frac{7}{25} = \frac{25}{25}+\frac{7}{25} = \frac{32}{25}$$

   Substituting back:

$$\tan\frac{\theta}{2} = \frac{-\frac{24}{25}}{\frac{32}{25}} = \frac{-24}{32} = -\frac{3}{4}$$

Final Answer:

$$\tan\frac{\theta}{2} = -\frac{3}{4}$$